2006

MS 221 MOCK EXAM

PART I

Instructions

(i) *You should attempt as many questions as you can in this part of the examination.*

(ii) Part 1 carries 72% of the available examination marks. Each question carries an indication of the number of marks that are allocated to it.

(iii) You should record your answers to each question in the answer book(s) provided. You are strongly advised to show all your working, including any rough working.

Question 1 - 6 marks

(a) Use the Fibonacci recurrence relation to show that

$$\frac{F_n}{F_{n+1}F_{n+2}} = \frac{1}{F_{n+1}} - \frac{1}{F_{n+2}} \qquad \text{for } n = 0, 1, 2\ldots \qquad [3]$$

(b) Hence use the method of telescoping cancellation to show that

$$\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{2}{3\times 5} + \frac{3}{5\times 8} + \ldots \frac{F_n}{F_{n+1}F_{n+2}} = 1 - \frac{1}{F_{n+2}} \qquad \text{for } n = 0, 1, 2\ldots \qquad [3]$$

Question 2 - 6 marks

This question concerns the curve with equation

$$9x^2 + 4y^2 - 36x + 8y + 4 = 0 \quad \text{(equation 1)}$$

(a) Show that this curve can be obtained from the conic with equation

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

by translation, and state the translation required. [3]

(b) Hence sketch the curve with equation 1, showing its axes of symmetry and the coordinates of its vertices. [3]

Question 3 - 6 marks

The isometry g is defined to be the reflection q_θ in the line $y = \frac{3x}{4}$ followed by the translation $t_{8,6}$.

(a) Show that g is a glide-reflection. [1]

(b) Give the rules for q_θ and $t_{8,6}$; hence determine the rule for g. [3]

(c) Determine the image of each of the points (0, 0), (0, 1), (1, 0) and (1, 1) under g. [2]

Question 4 - 5 marks

From a group of 9 people, of whom 5 are men and 4 are women, 5 are to be selected to form a committee.

(a) In how many different ways may this selection be made? [2]

(b) In what proportion of this total number of possible selections will women outnumber men on the committee? [3]

Question 5 - 7 marks

(a) Find the fixed points of the function $f(x) = 2x(1 - x)$ where $x \in \mathbb{R}$. [2]

(b) Draw a sketch of the graph of f, together with the line $y = x$, showing the fixed points, and hence explain whether each fixed point is attracting or repelling. [2]

(c) Use the gradient criterion to find an interval of attraction for one of the fixed points. [3]

Question 6 - 6 marks

Express the matrix M below in the form QDQ^{-1}, where D is a diagonal matrix (you should calculate the matrix Q^{-1} explicitly):

$$M = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$$ [6]

Question 7 – 4 marks

Differentiate the following functions. In each case, state which of the principal rules of calculus you are using.

(a) $k(x) = \ln(\arcsin(x)) \quad (0 < x < 1)$ [2]

(b) $g(t) = t^3 \cos(3t)$ [2]

Question 8 - 5 marks

(a) Using integration by parts, find the indefinite integral

$$\int x^3 \ln(3x)\,dx \quad (x > 0).$$ [2]

(b) Using the substitution $u = \ln(x)$, or otherwise, find the indefinite integral

$$\int \frac{\sec^2(\ln(x))}{x}\,dx.$$ [3]

Question 9 - 4 marks

(a) Find the volume of revolution obtained when the region under the graph of

$y = \frac{1}{x}$, from x = 1 to x = k (where k > 1 is a constant), is rotated about the x-axis. [3]

(b) Hence show that this volume tends to a finite limit as $k \to \infty$. [1]

Question 10 - 5 marks

(a) Use appropriate results from the Handbook to find the first three non-zero terms of the Taylor series about $x = 0$ for the function

$$f(x) = \sin(x)\cos(x).$$ [2]

(b) By differentiating term by term your answer to (a), show that the Taylor polynomial of degree 4 for $f'(x)$ is the same as that for cos(2x), and explain why you would expect this to be the case. [3]

Question 11 - 5 marks

Let $z = 8\exp\left(\frac{i\pi}{3}\right)$.

(a) Express z and $\bar{z}$ in Cartesian form, where $\bar{z}$ is the complex conjugate of z. [3]

(b) Hence, find in Cartesian form $z + \bar{z}$ and $z\bar{z}$. [2]

Question 12 - 4 marks

(a) Show that 658 324 719 is divisible by 9 but not divisible by 18. [2]

(b) Find a number x in $\mathbb{Z}_{15}$ such that $x \times_{15} 4 = 5$. [1]

(c) Give an example of a number x in $\mathbb{Z}_{15}$ which has no multiplicative inverse. [1]

Question 13 - 4 marks

Consider the group $(G, *)$ whose Cayley table is given below.

$*$	p	q	r	s
p	r	s	p	q
q	s	r	q	p
r	p	q	r	s
s	q	p	s	r

(a) Which element is the identity element of $(G, *)$? [1]

(b) Write down all the self-inverse elements of $(G, *)$. [1]

(c) Is $(G, *)$ Abelian ? Briefly explain your answer. [1]

(d) To which of the groups listed in the Handbook is $(G, *)$ isomorphic ? [1]

Question 14 - 5 marks

Here are two statements about integers n and m, only one of which is true.

(A) If n and m are both even then $n^2 - m^2$ is even.

(B) If $n^2 - m^2$ is even then n and m are both even.

(a) Which of these statements is false ? [1]

(b) Prove that the statement you have identified in (a) is false. [3]

(c) What is the name of the style of proof you have used in (b) ? [1]

PART II

Instructions

(i) *You should attempt not more than* **TWO** *questions from this part of the examination.*

(ii) *Each question in this part carries 14% of the marks.*

(iii) *You may answer the questions in any order. Write your answers in the answer book(s) provided, beginning each question on a new page.*

(iv) *Show all your workings.*

Question 15

(a) Find a closed form for the sequence given by the following recurrence system:

$u_0 = 1,\ u_1 = 2,\ \ u_{n+2} = -4u_{n+1} - 4u_n \qquad (n = 0, 1, 2, \ldots)$ [6]

(b) Show that this sequence satisfies the identity

$u_{n+1}u_{n-1} - u_n^{\,2} = -4^{n+1} \qquad (n = 1, 2, 3, \ldots)$ [4]

(c) Show that the sequence also satisfies the identity

$$\frac{u_{n+1}}{u_n} = \frac{2+4n}{1-2n} \qquad (n = 0, 1, 2\ldots)$$

and deduce that $\dfrac{u_{n+1}}{u_n} \to -2$ as $n \to \infty$. [4]

Question 16

(a) Find, in the form $\boldsymbol{x} \mapsto \boldsymbol{Ax} + \boldsymbol{a}$ where $\boldsymbol{A}$ is a 2x2 matrix and $\boldsymbol{a}$ is a vector with two components, the rule of the affine transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ that maps the points (0, 0), (1, 0) and (0, 1) to the points (3, 2), (5, 1) and (6, 0) respectively. [3]

(b) Hence find the area of the triangle with vertices (3, 2), (5, 1) and (6, 0). [1]

(c) Show that the matrix $\boldsymbol{A}$ that you found in part (a) is self-inverse (that is, that $\boldsymbol{A} = \boldsymbol{A}^{-1}$). [2]

(d) Let $\boldsymbol{x_1} = \boldsymbol{Ax_0} + \boldsymbol{a}$, where $\boldsymbol{A}$ and $\boldsymbol{a}$ are the matrix and vector from part (a) and $\boldsymbol{x_0}$ is an arbitrary two-component vector. Use the result of part (b) to show that $\boldsymbol{x_0} = \boldsymbol{A}(\boldsymbol{x_1} - \boldsymbol{a})$. [3]

(e) Use the result of part (d) to find, in the form $\boldsymbol{x} \mapsto \boldsymbol{Bx} + \boldsymbol{b}$ where $\boldsymbol{B}$ is a 2x2 matrix and $\boldsymbol{b}$ is a two-component vector, the rule of the affine transformation f^{-1}. [2]

(f) Hence find the images under f of the lines $y = x$ and $y = -x$. [3]

Question 17

Use the graph-sketching strategy of Chapter C1 to sketch the graph of the real function $f(x)=\dfrac{2x+1}{2(3x+1)(x-1)}$. [14]

Question 18

(a) Prove, using mathematical induction, that

$$1^3 + 2^3 + \ldots + n^3 = (1 + 2 + \ldots + n)^2$$

for all positive integers n.

[You may assume, without proof, the formula $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$.] [8]

(b) Use Fermat's Little Theorem to find the remainder when 221^{2004} is divided by 19.

[6]

[END OF QUESTION PAPER]